Int. J. Solids Structures, 1967, Vol. 3, pp. 1013 to 1021. Pergamon Press Ltd. Printed in Great Britain

DEPENDENCE OF STRESS ON POISSON'S RATIO IN PLANE ELASTICITY*

J. DUNDURS

Northwestern University, Evanston, Illinois

Abstract—The conditions in plane elasticity under which stress is independent of Poisson's ratio are well known. In the present note, the explicit dependence of stress on Poisson's ratio is derived for the case when these conditions are not met because of unbalanced forces on internal boundaries. The result shows that knowledge of stress in two models with different Poisson's ratios is sufficient for the determination of stress in a prototype of any Poisson's ratio. Extensions regarding body forces and dislocation, temperature and residual stresses are indicated.

INTRODUCTION

WITHIN the framework of linear theory, stress caused in an elastic and isotropic body by given surface tractions can depend only on a dimensionless combination of elastic constants, such as Poisson's ratio. The dependence of stress on Poisson's ratio is a complicating factor in both theory and experimental work, and occasional investigations have dealt with this question. Among the more recent contributions are the papers by Knops [1] and Sternberg and Muki [2]. However, it appears that an explicit dependence of stress on Poisson's ratio can be derived only for plane elasticity.

Among the various propositions of plane elasticity, the *Michell result* [3] has played one of the most prominent roles in applications. If a body is loaded by specified surface tractions and there are no body forces, the theorem by Michell asserts that the stress in the body is independent of Poisson's ratio, provided either the body is singly connected, or the tractions over each interior boundary give no net force. The encouragement this result has given to work in photoelasticity is well known, but sometimes it is equally expedient in approximate methods.

There are, of course, complications in experimental work when the stress depends on Poisson's ratio because of unbalanced forces on internal boundaries, and at the same time the model cannot be made of a material with a Poisson's ratio equal to that of the prototype. One method for dealing with this problem is due to Filon [4, 5]. His technique requires that two experiments be performed; one with the given loading on the body, the other with the body free of applied tractions, but dislocated in the sense of Volterra. To introduce Volterra dislocations in a model is not easy, and one gets the impression that, generally, it has been more convenient to take refuge behind the estimates of Filon and Bickley [6]. However, the specific examples used by Filon and Bickley to probe the effect of Poisson's ratio involved essentially a single geometry, and their results, indicating but a mild dependence, are not universally true. Indeed, it is not difficult to invent counterexamples showing that Poisson's ratio can have a sizeable effect. The simplest of these may be the body in the outline of a bucket, loaded as shown in Fig. 1. If the vertical arms are made sufficiently

* The results reported here were obtained in the course of research supported by Advanced Research Projects Agency of the U. S. Department of Defence through the Material Research Center of Northwestern University.

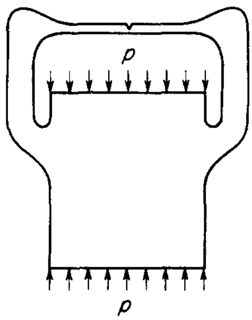

FIG. 1

stiff, the strain in the thin horizontal bar will roughly be equal to the transverse strain in the block. The stress in the bar is then νp, and it can range anywhere from 0 to $p/2$, depending upon the particular material. The stress in the thin bar may be manipulated further by changing the length of the bar, or the whole point can be driven to the extreme by notching the bar, so that the largest stress in the body becomes $k\nu p$, with $k \gg 1$.

In spite of the fact that plane elasticity is a well developed field, it seems that an interesting and simple result regarding the dependence of stress on Poisson's ratio has been overlooked in the past. The main purpose of this note is to show that, in a body subjected to given surface tractions, *stress can be found for any Poisson's ratio, provided it is known for two specific values of this ratio.* The proof is based on the properties of the stress field for a point load in the elastic whole plane and on the Michell result. As the results remain valid for concentrated forces, there are also immediate extensions to stresses caused by body forces and inelastic strains.

The usefulness of the present results in experimental work would depend on the availability of two materials with significantly different Poisson's ratios. In numerical methods, however, there are no such difficulties, and stresses computed for the extremes of $\nu = 0$ and $\nu = \frac{1}{2}$ will yield the answer for any Poisson's ratio.

DEPENDENCE OF STRESS ON POISSON'S RATIO IN PLANE ELASTICITY

It will be convenient in the subsequent discussion to reserve subscripts for the tensorial indices ($i, j = 1, 2$; and it is summed over a repeated index), and to use superscripts for labels.

The reasoning is based largely on three properties of the stress field for a concentrated force in the elastic whole plane. In terms of the polar coordinates ρ and α, shown in Fig. 2, the Airy stress function is in this case

$$U^{\infty} = \frac{P}{2\pi(\varkappa+1)}[-(\varkappa+1)\rho\alpha \sin\alpha + (\varkappa-1)\rho \log\rho \cos\alpha], \tag{1}$$

where P is the magnitude of the applied force P_i. Denoting Poisson's ratio by ν,

$$\varkappa = \frac{3-\nu}{1+\nu} \tag{2}$$

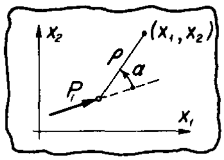

FIG. 2

for *plane stress*,

$$\varkappa = 3-4\nu \tag{3}$$

for *plane strain*, and we have $1 \leq \varkappa \leq 3$.

It may be observed that U^{∞} is of the form

$$B(x_1, x_2; \varkappa) = \frac{\varkappa f(x_1, x_2)+g(x_1, x_2)}{\varkappa+1}. \tag{4}$$

Here $f(x_1, x_2)$ and $g(x_1, x_2)$ are functions of position that do not contain $\varkappa$, or

$$\frac{\partial f}{\partial \varkappa} = \frac{\partial g}{\partial \varkappa} = 0. \tag{5}$$

Functions that are of the form given by (4) will be called bilinear in $\varkappa$, or of form B. Clearly, all functions independent of $\varkappa$ can be put in form B.

Denoting the stress derived from $U = -\frac{1}{2}\pi\, \rho\alpha \sin\alpha$ by $a_{ij}(x_1, x_2)$ and that from $U = \frac{1}{2}\pi\, \rho \log \rho \cos\alpha$ by $b_{ij}(x_1, x_2)$, the total stress in the whole elastic plane is

$$\sigma_{ij}^{\infty} = \frac{P}{\varkappa+1}\{\varkappa[a_{ij}(x_1, x_2)+b_{ij}(x_1, x_2)]+a_{ij}(x_1, x_2)-b_{ij}(x_1, x_2)\}. \tag{6}$$

The detailed nature of the six functions a_{ij} and b_{ij} in (6) is of no relevance, but it is important for our purposes to observe the following:

(a) The stress σ_{ij}^{∞} is bilinear in $\varkappa$.

(b) The tractions derived from a_{ij} give a net force of unit magnitude on all closed contours surrounding the point of application of the force. Of course, there is no net force from a_{ij} on contours not encircling this point.

(c) The tractions from b_{ij} are self-equilibrated on *all* closed contours.

The boundary curves of the multiply connected body, shown in Fig. 3, will be denoted by $C^0, C^1, C^2 \ldots C^k, \ldots C^n$, with C^0 reserved for the outer boundary. The arc-length s^k, measured along C^k from some point on the curve, can be used as a parameter to specify the position of a given point on C^k.

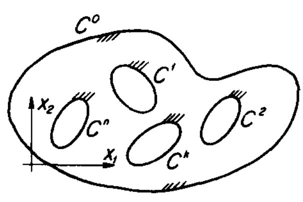

FIG. 3

The body is subjected to specified surface tractions t_i^k on all C^k. It is expedient to set the tractions proportional to a single scalar factor q which specifies the general intensity of loading. Thus,

$$t_i^k = qF_i^k(s^k), \tag{7}$$

and a change in q amounts only to raising or lowering the level of loading, while the tractions remain similar.

The net force on C^k is

$$P_i^k = \oint_{C^k} t_i^k \, \mathrm{d}s^k = q\oint_{C^k} F_i^k(s^k)\, \mathrm{d}s^k = qp_i^k. \tag{8}$$

If all p_i^k vanish, then according to the Michell result, the stress in the body is independent of Poisson's ratio, and it can be put in form B. The object is to show, however, that the stress in the body is bilinear in $\varkappa$ even when some p_i^k do not vanish, and the stress depends on Poisson's ratio.

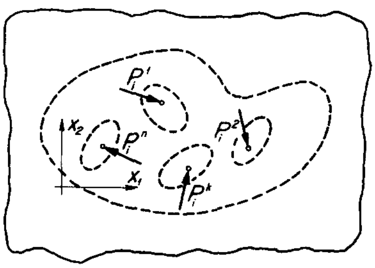

FIG. 4

For that purpose take the elastic whole plane, sketch in the outline of the given body, as shown in Fig. 4, and subject the infinite domain to the *concentrated forces* $P_i^1, P_i^2, \ldots P_i^k, \ldots P_i^n$. These forces are computed from formula (8), and they are applied at arbitrary points inside the outlines of $C^1, C^2, \ldots C^k, \ldots C^n$. The particular point chosen inside, say, C^k does not matter, because P_i^k needs not be equipollent to the system of elemental forces $t_i^k \, \mathrm{d}s^k$ acting on C^k in the original problem. It also may be noted that P_i^0 is *not* applied to the elastic whole plane. Using (6) and superposing, the stress in the infinite domain is then

$$\sigma_{ij}^{\infty} = \frac{q}{\varkappa+1}\sum_{k=1}^{n} p^k[\varkappa(a_{ij}^k+b_{ij}^k)+a_{ij}^k-b_{ij}^k]. \tag{9}$$

Here a_{ij}^k and b_{ij}^k are the counterparts of a_{ij} and b_{ij} in (6); they are the contributions of the individual forces P_i^k.

The traction t_i is computed from stress σ_{ij} by the formula

$$t_i = \sigma_{ij}n_j, \tag{10}$$

where n_j is the unit normal to the boundary directed out of the material. If we think of stress σ_{ij}^{∞} as *existing in the given body*, we can obtain from (10) the tractions $(t_i^k)^{\infty}$ that would have to be applied to the boundaries $C^0, C^1, \ldots C^k, \ldots C^n$ of the given body in order to achieve this state of stress. Denoting with G_i^{kl} and H_i^{kl} the tranctions on C^k that are due to the individual terms $p^l a^l_{ij}$ and $p^l b^l_{ij}$, respectively, in (9), we have

$$(t^k_i)^\infty = \frac{q}{\varkappa+1}\sum_{l=1}^{n}[\varkappa(G^{kl}_i+H^{kl}_i)+G^{kl}_i-H^{kl}_i]. \tag{11}$$

If the stress in the body produced by the specified tractions t^k_i is σ_{ij}, we can change σ^∞_{ij} to σ_{ij} by subjecting the body, in addition to $(t^k_i)^\infty$, to the tractions

$$(t^k_i)^R = t^k_i-(t^k_i)^\infty. \tag{12}$$

The tractions $(t^k_i)^R$, applied to all C^k, may be called the residual loading of the body. Substituting (7) and (11) into (12), the result is

$$(t^k_i)^R = \frac{q}{\varkappa+1}\left\{\varkappa\left[F^k_i-\sum_{l=1}^{n}(G^{kl}_i+H^{kl}_i)\right]+F^k_i-\sum_{l=1}^{n}(G^{kl}_i-H^{kl}_i)\right\}. \tag{13}$$

On basis of property (c) for the stress in the elastic whole plane caused by a concentrated force,

$$\oint_{C^k} H^{kl}_i\,\mathrm{d}s^k = 0, \tag{14}$$

and, consequently, the two sets of tractions

$$F^k_i-\sum_{l=1}^{n}(G^{kl}_i+H^{kl}_i), \tag{15}$$

$$F^k_i-\sum_{l=1}^{n}(G^{kl}_i-H^{kl}_i), \tag{16}$$

individually give no net forces on the boundaries $C^0, C^1, \ldots C^k, \ldots C^n$. Therefore, if the body is loaded by either the tractions (15) or (16), the stress in the body is independent of Poisson's ratio in consequence of Michell's result. This allows us to conclude that the stress σ^R_{ij}, produced by the action of $(t^k_i)^R$ on the boundaries, must be of the same form in $\varkappa$ as (13), or

$$\sigma^R_{ij} = \frac{q}{\varkappa+1}[\varkappa c_{ij}(x_1,x_2)+d_{ij}(x_1,x_2)]. \tag{17}$$

Since the total stress in the body is

$$\sigma_{ij} = \sigma^\infty_{ij}+\sigma^R_{ij}, \tag{18}$$

it follows finally from (9) and (17) that, *regardless how a body is loaded by specified surface tractions, the stress is bilinear in* $\varkappa$, or

$$\sigma_{ij} = \frac{q}{\varkappa+1}[\varkappa\Phi_{ij}(x_1,x_2)+\varphi_{ij}(x_1,x_2)]. \tag{19}$$

It may be noted that (19) is valid also when the loading includes concentrated forces or moments applied at the boundaries or interior points. Substituting for $\varkappa$ and absorbing constants that do not matter into the functions, (19) reduces for *plane stress* to

$$\sigma_{ij} = q[\Psi_{ij}(x_1,x_2)+\nu\psi_{ij}(x_1,x_2)], \tag{20}$$

and for *plane strain* to

$$\sigma_{ij} = \frac{q}{1-\nu}[\Omega_{ij}(x_1, x_2) + \nu\, \omega_{ij}(x_1, x_2)]. \tag{21}$$

STRESSES IN BODIES WITH DIFFERENT POISSON'S RATIOS

Suppose that we wish to compare stresses in two geometrically similar bodies subjected to similar surface tractions, but having different Poisson's ratios. It is not interesting to become involved in scale effects because of different sizes and, therefore, the bodies will be taken as congruent. To distinguish between the two bodies we shall use superscripts 1 and 2. The point of the derivation which led to (19) was that the functions Φ_{ij} and φ_{ij} are independent of Poisson's ratio. Therefore, from (19)

$$\sigma_{ij}^1 = \frac{q^1}{\varkappa^1+1}[\varkappa^1 \Phi_{ij}(x_1, x_2) + \varphi_{ij}(x_1, x_2)], \tag{22}$$

$$\sigma_{ij}^2 = \frac{q^2}{\varkappa^2+1}[\varkappa^2 \Phi_{ij}(x_1, x_2) + \varphi_{ij}(x_1, x_2)]. \tag{23}$$

Also the expressions (20) and (21) would carry over in a similar fashion.

It is seen at a glance from (22) and (23) that the stress fields in the two bodies can never be made the same by simply adjusting the level of loading. However, the results also show that, if the stress is known for two Poisson's ratios, the functions Φ_{ij} and φ_{ij} can be computed from (22) and (23). Therefore, the stress is then known for all Poisson's ratios. Also, the identical result can be claimed if the stress is known for both plane stress and strain in the same material.

Finally it may be instructive to consider a prototype (superscript 0) and two models (superscripts 1 and 2) with different Poisson's ratios. If we make

$$q^1 = \frac{\varkappa^1+1}{\varkappa^0+1} q^0, \tag{24}$$

$$q^2 = \frac{\varkappa^2+1}{\varkappa^0+1} q^0, \tag{25}$$

then from (19)

$$\sigma_{ij}^0 = \frac{q^0}{\varkappa^0+1}(\varkappa^0 \Phi_{ij} + \varphi_{ij}), \tag{26}$$

$$\sigma_{ij}^1 = \frac{q^0}{\varkappa^0+1}(\varkappa^1 \Phi_{ij} + \varphi_{ij}), \tag{27}$$

$$\sigma_{ij}^2 = \frac{q^0}{\varkappa^0+1}(\varkappa^2 \Phi_{ij} + \varphi_{ij}). \tag{28}$$

It is seen, therefore, that the stress at any point in the prototype can be obtained from the stresses at the same points in the models by means of the linear inter- or extrapolation shown in Figs. 5. If one limits himself to plane stress, it follows from (20) that the load parameters for the models can be kept the same as that for the prototype, and the linear interpolation done versus the Poisson's ratio.

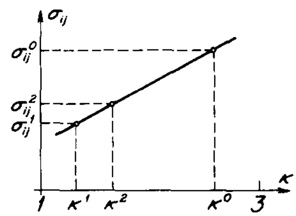

FIG. 5

EXTENSIONS

Results, similar to those for loading by surface tractions, also can be obtained for body forces, dislocations and stresses caused by temperature changes or inelastic strains. In the last three cases, it is understood that the boundaries of the body are free of external forces, except for the necessary end-constraints in plane strain. The proof of these results is not difficult, and it will be left as an exercise.

Body forces

If there are body forces in addition to possible prescribed surface tractions, the stress is of the form

$$\sigma_{ij} = \frac{q}{\varkappa+1}[\varkappa f_{ij}(x_1, x_2)+g_{ij}(x_1, x_2)], \tag{29}$$

where again q is a factor specifying the intensity of loading.

Volterra and singular dislocations

The stress, produced by dislocating a multiply connected body in the sense of Volterra, is

$$\sigma_{ij} = \frac{Gb}{\varkappa+1} f_{ij}(x_1, x_2), \tag{30}$$

where G is the shear modulus, and b specifies the magnitude of dislocation. It is seen from (30) that the elastic constants enter the stress merely as a scale factor. Therefore, knowledge of the stress field for one material is enough to find it for any other material. Equation (30) also is true for the singular dislocations of the edge-type.

Temperature stresses

Denoting by α the coefficient of thermal expansion, and by τ a factor that specifies the level of temperature from the reference state, the stress is of the form

$$\sigma_{ij} = \frac{G(1+\eta)\alpha\tau}{\varkappa+1} f_{ij}(x_1, x_2). \tag{31}$$

Here $\eta = 0$ for plane stress, and $\eta = \nu$ for plane strain. Also in this case the elastic constants appear only as a scale factor.

Residual stresses

Writing generalized Hooke's law for an elastic isotropic solid in the form

$$\varepsilon_{pq} = \varepsilon^e_{pq} + \varepsilon^0_{pq} = \frac{1}{2G}\left(\sigma_{pq} - \frac{\nu}{1+\nu}\sigma_{rr}\delta_{pq}\right) + \varepsilon^0_{pq}, \qquad (p, q, r = 1, 2, 3), \tag{32}$$

the term ε^0_{pq} is seen to represent the strain that elements of the material would suffer in the absence of stress. Sometimes ε^e_{pq} is called the elastic strain and ε^0_{pq} the inelastic strain or eigenstrain. The simplest example of eigenstrain is the strain caused by change in temperature, for which $\varepsilon^0_{pq} = \alpha T\delta_{pq}$. In martensitic transformations, to name another example, ε^0_{pq} is nearly deviatoric. By necessity, ε^0_{pq} is symmetric but, generally, it is incompatible. As the total strain ε_{pq} in (32) must be compatible, it is precisely the incompatibility of the eigenstrains that gives rise to residual stresses in a body.

For plane deformations we must have $\varepsilon^0_{23} = \varepsilon^0_{31} = 0$. If (32) is specialized to plane deformations, the result is

$$\varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{3-\varkappa}{4}\sigma_{kk}\delta_{ij}\right) + \varepsilon^0_{ij} + \eta\varepsilon^0_{33}\delta_{ij}, \qquad (i, j, k = 1, 2). \tag{33}$$

Here again $\eta = 0$ for plane stress, and $\eta = \nu$ for plane strain. Finally, the stress can be shown to be of the form

$$\sigma_{ij} = \frac{G}{\varkappa+1}[e^0 f_{ij}(x_1, x_2) + \eta\varepsilon^0_{33} g_{ij}(x_1, x_2)], \tag{34}$$

where e^0 specifies the intensity of the in-plane components of eigenstrain, or $\varepsilon^0_{11}, \varepsilon^0_{12}, \varepsilon^0_{22}$.

REFERENCES

[1] R. J. KNOPS, On the variation of Poisson's ratio in the solution of elastic problems. *Q. Jl Mech. appl. Math.* **11**, 326 (1958).
[2] E. STERNBERG and R. MUKI, Note on the expansion in powers of Poisson's ratio of solutions in elastostatics. *Archs ration. Mech. Analysis* **3**, 229 (1959).
[3] J. H. MICHELL, On the direct determination of stress in an elastic solid, with application to the theory of plates. *Proc. Lond. math. Soc.* **31**, 100 (1899).
[4] L. N. G. FILON, On stresses in multiply-connected plates. *Rep. 89 Meet., Br. Ass. Advmt. Sci.* 305 (1921).
[5] E. G. COKER and L. N. G. FILON, *A Treatise on Photo-elasticity*. Cambridge University Press (1931).
[6] W. G. BICKLEY, The distribution of stress round a circular hole in a plate. *Phil. Trans. R. Soc.* **A227**, 383 (1928).

(*Received* 10 *May* 1967)

Résumé—Les conditions en élasticité plane dans lesquelles la tension est indépendante du coefficient de Poisson sont bien connues. Dans la présente note, la dépendance explicite de la tension sur le coefficient de Poisson dérive du cas où ces conditions ne se présentent pas par suite de forces non équilibrées aux limites internes. Le résultat montre que la connaissance de la tension dans deux maquettes avec des coefficients de Poisson différents suffit pour établir la tension dans un prototype à coefficient de Poisson quelconque. Des résultats supplémentaires relatifs les forces du corps et la dislocation, la température et les tensions résiduelles sont indiqués.

Zusammenfassung—Die Bedingungen, unter denen die Spannung in ebenen Elastizitätsproblemen unabhängig von der Querdehnungszahl ist, sind wohlbekannt. In dieser Arbeit wird die Abhängigkeit der Spannung von der Querdehnungszahl für den Fall abgeleitet, wenn diese Bedingungen verletzt sind, weil die Vektorensummen der an inneren Grenzen angreifenden Oberflächenkräfte nicht verschwinden. Das Ergebnis zeigt, dass die Kenntnis der Spannung in zwei Modellen mit verschiedenen Querdehnungszahlen dafür hinreichend ist, die Spannung bei beliebiger Querdehnungszahl zu bestimmen. Erweiterungen für Massenkräfte, Versetzungen sowie Wärme- und Eigenspannungen werden angedeutet.

Абстракт—В плоской теории упругости хорошо известны такие условия, под влиянием которых напряжение является независимым от коэффициента Пуассона. В настоящей заметке выводится в явном виде зависимость между напряжением и коэффициентом Пуассона для случая, когда эти условия не существуют в следствие неуравновешенных сил на внутренних контурах. Результат указует на то, что знание напряжений в двух моделях с разными коэффициентами Пуассона, оказуется достаточным для определения напряжения в опытном образце слюбым коэффициентом Пуассона. Указывается на расширения темы, касающейся влияния массовых сил, дислокации, температуры и остаточных напряжений.